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Numerical Methods for Flood Routing

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SONY



Before and after

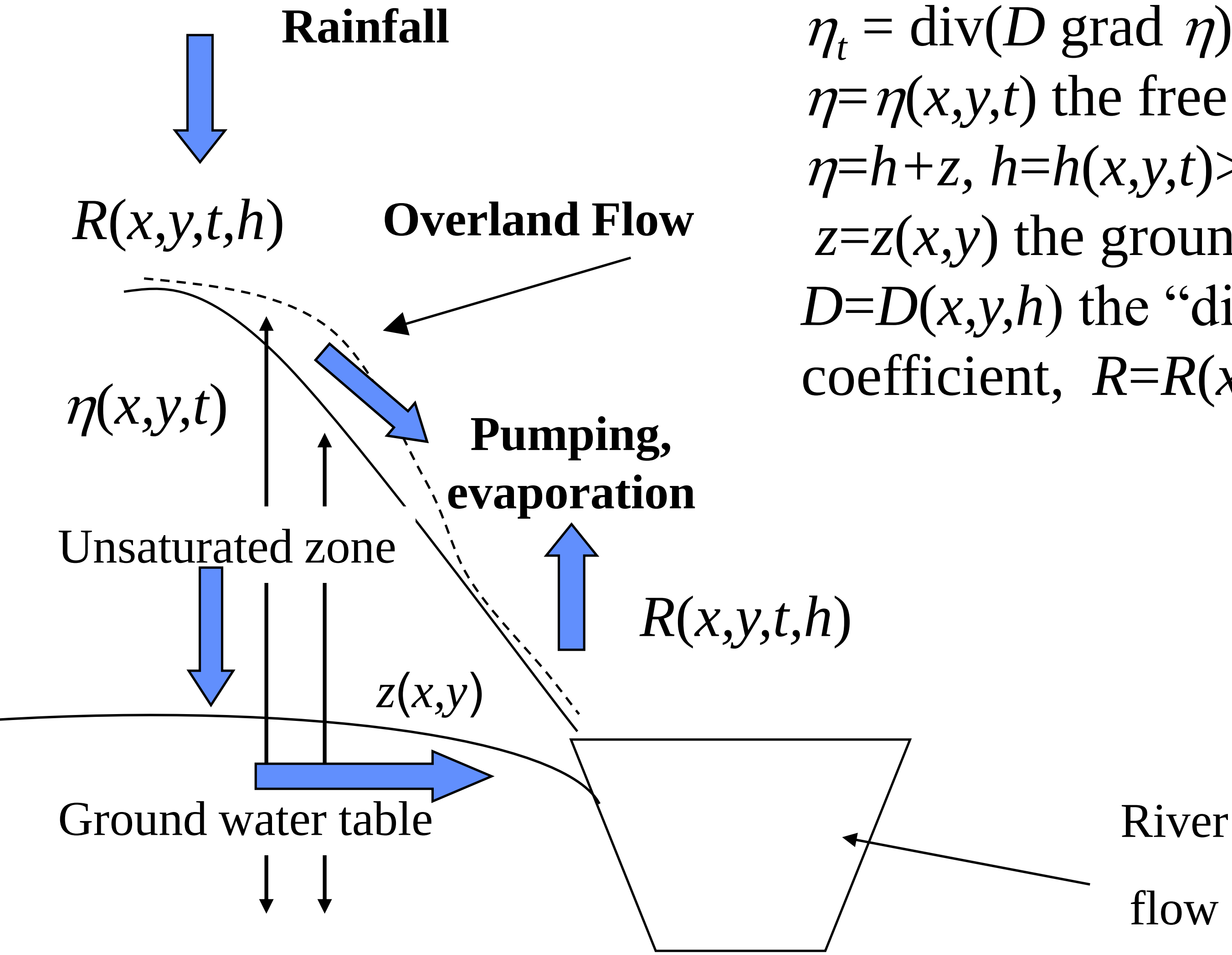
The numerical model is designed for water systems comprising large canal networks. The mathematical formulation is based on the analogy of a **porous medium** characterized by a permeability depending on parameters and directions of the canal network. We combine the **"continuous medium approach**” with **the non-negative stable numerical algorithm**.

We simulate the flood evolution in the lower Chao-Praya river basin (located in the eastern areas of Bangkok) and demonstrate the computational e efficiency of the proposed method.

Flood flow

Diffusion wave approach

$\eta_t = \mathrm{div}(D \text{ grad } \eta) + R$,
$\eta = \eta(x,y,t)$ the free surface level,
$\eta = h + z$, $h = h(x,y,t) > 0$ the flow depth,
$z = z(x,y)$ the ground elevation,
$D = D(x,y,h)$ the “diffusion” coefficient, $R = R(x,y,t,h)$ the source



Diffusion wave equation

$$w_h \frac{\partial \eta}{\partial t} = \mathrm{div}\left(\frac{\alpha}{\sqrt{|\nabla \eta|}} \nabla \eta\right) + T,$$

$$\frac{\partial}{\partial s} D \frac{\partial \eta}{\partial s}$$

$$D = wR^{2/3} \Big/ \left(n_f \left| \frac{\partial \eta}{\partial x} \right|^{1/2} \right)$$ -1D river flow

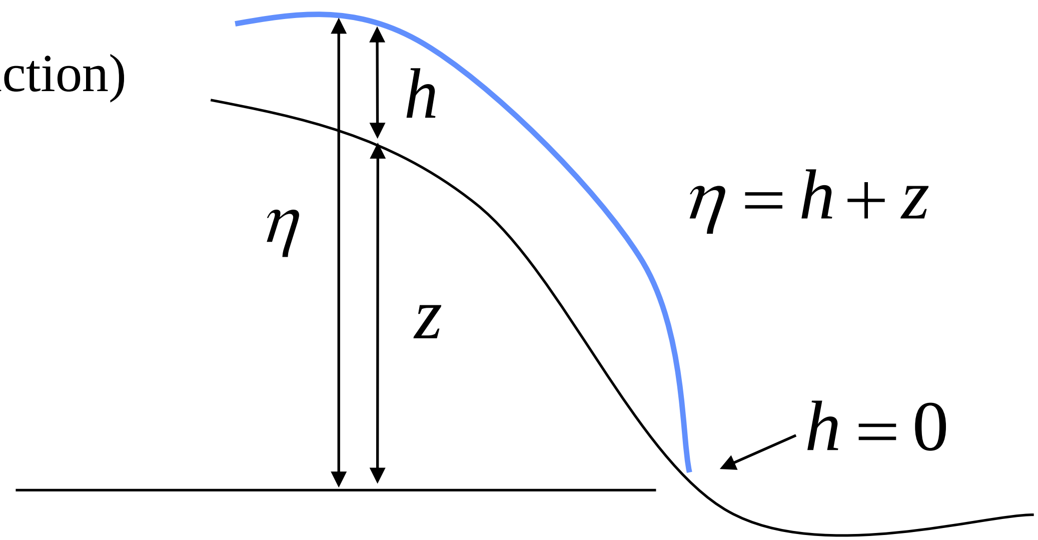
$$D = h^{5/3} / (n_f \,|\, \mathrm{grad}\, \eta \,|^{1/2})$$ -2D overland flow

$$\frac{\partial}{\partial x} D \frac{\partial \eta}{\partial x} + \frac{\partial}{\partial y} D \frac{\partial \eta}{\partial y}$$

n_f - Manning coefficient (friction)

R - hydraulic radius

w - cross sectional area



Flood models based on the shallow water equations

Separate treatment of the river and the surface flows/**1D Saint Venant + 2D shallow water equations.**

-Sophisticated mathematical formulation employing the internal boundary conditions, the confluence conditions, etc.

-Complicated data structures

-Tedious iterative techniques to couple the river and the surface flow

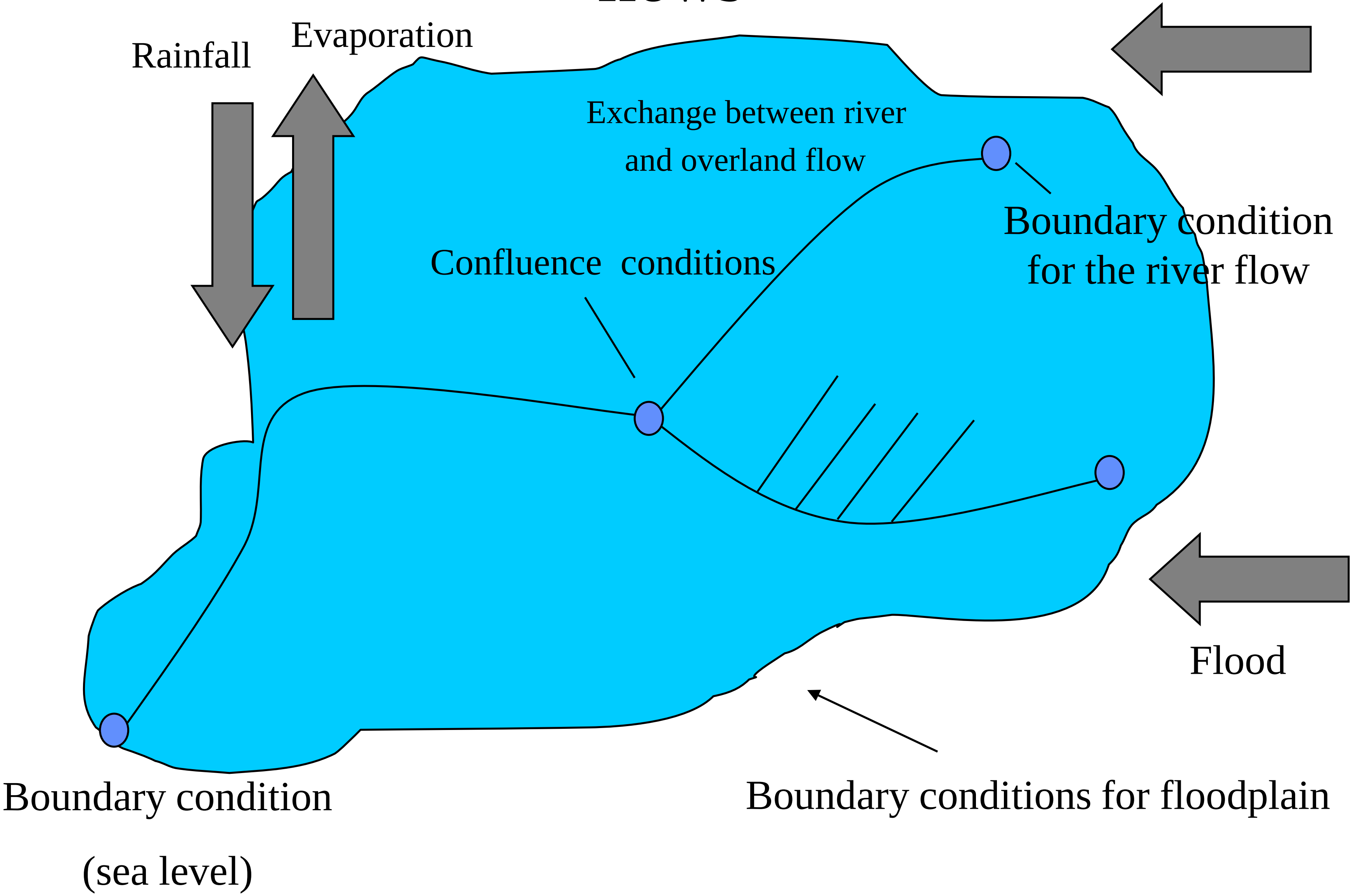
-Special control of possible negative components of the solution

Uniform treatment of the river and the surface flows/**2D shallow water equations for both the river and the surface flows**

-High spatial resolution required in the regions nearby the river

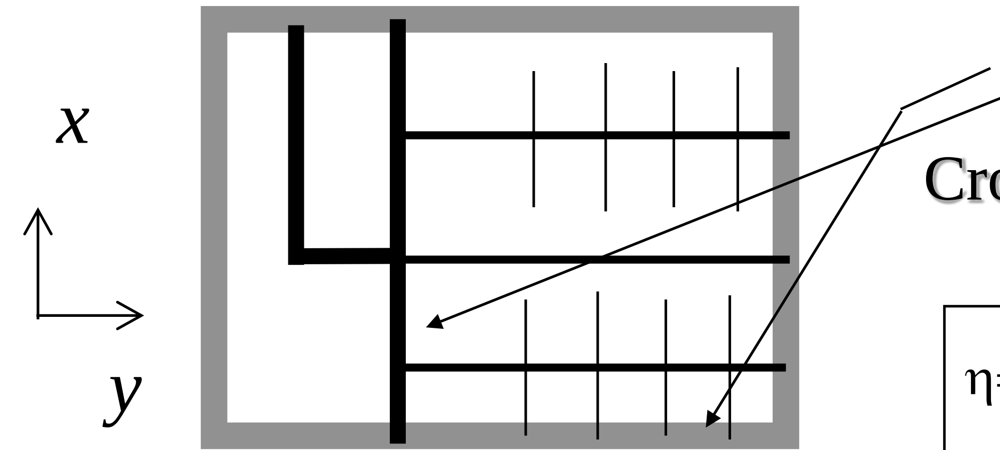
-Special control of possible negative components of the solution in the regions with the small water depth.

Separate treatment of the river and the surface flows

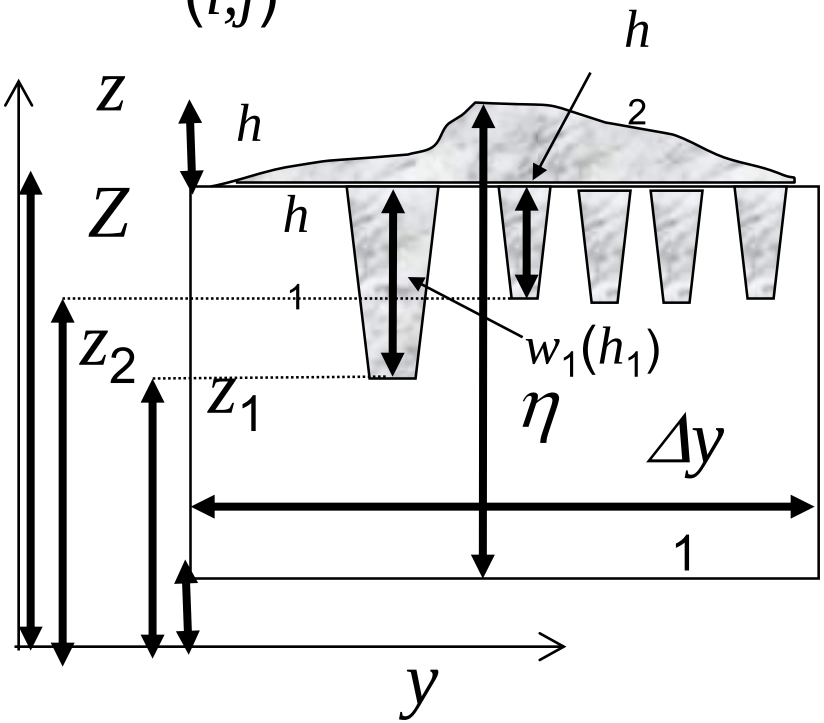


Uniform treatment of the river/overland flow

"Channelized" cell of the 2D grid



(*i*,*j*)



Cross-sectional area of the flow is characterized by

$\eta = h + z_0$ the surface flow level

$h = \eta - Z$ the surface flow depth

$h_i - z_i$ the water depth for canals of a group i orthogonal to the y-axis

m^y_i - number of canals of a group i orthogonal to the y-axis

$W^y_i(h_i) = w_i(h_i)/\Delta y$ the vertical cross-sectional area of the canals of a group i orthogonal to the y-axis

$h\Delta y$ the cross-sectional area of the surface flow

The modified diffusion wave equation is then given by

$$w_h \frac{\partial \eta}{\partial t} = \frac{\partial}{\partial x}[(D_R^x + D_S^x)\frac{\partial \eta}{\partial x}] + \frac{\partial}{\partial y}[(D_R^y + D_S^y)\frac{\partial \eta}{\partial y}] + T.$$

surface

river

$$w_h = \begin{cases} \sum_L \sum_p m_p W_p^L, \text{if } \eta < Z, \\ 1, \qquad \text{otherwise.} \end{cases}$$

$W^L{}_p = w^L{}_p/\Delta L$, $L=x$ or $L=$y, $w^L{}_p$ denotes the vertical cross-sectional area of the canals of the group p with an average bottom level z_p, a cross-sectional area w_p along the L direction. A number of canals at the point (x,y) is denoted by m_p. We recall that $\eta \equiv \eta(x,y,t)$ denotes the water level, $\eta = h + Z$, $h = h(x,y,t)$ the water depth. $Z = Z(x,y)$ is the ground level

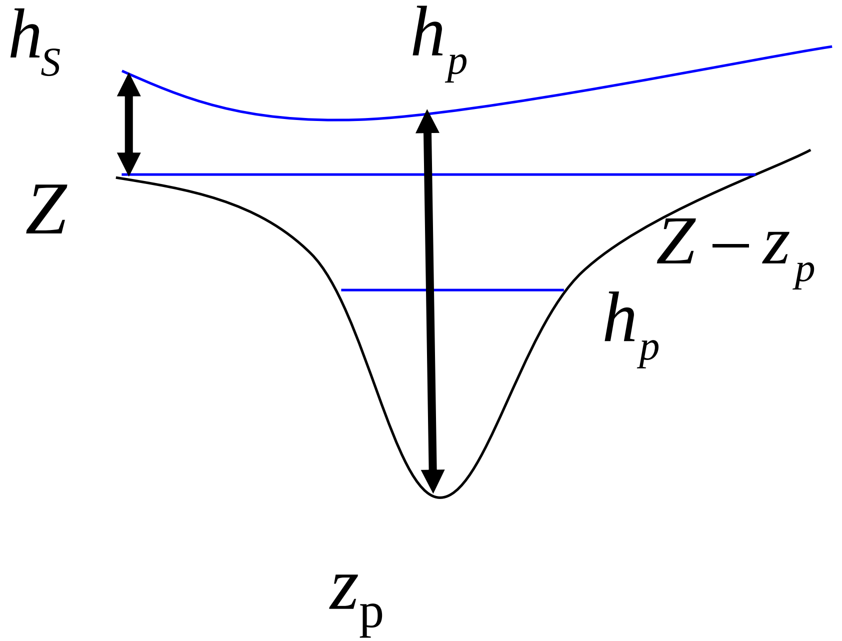
The diffusion coefficients corresponding to the canal and the overland flow are

$$D_R^L = \sum_p m_p \frac{W_p^L(h_p)[R(h_p)]^{2/3}}{n_p \left| \frac{\partial \eta}{\partial L} \right|^{1/2}}, \quad D_S^L = \frac{h_S^{5/3}}{n_s |\mathrm{grad}\,\eta|^{1/2}},$$

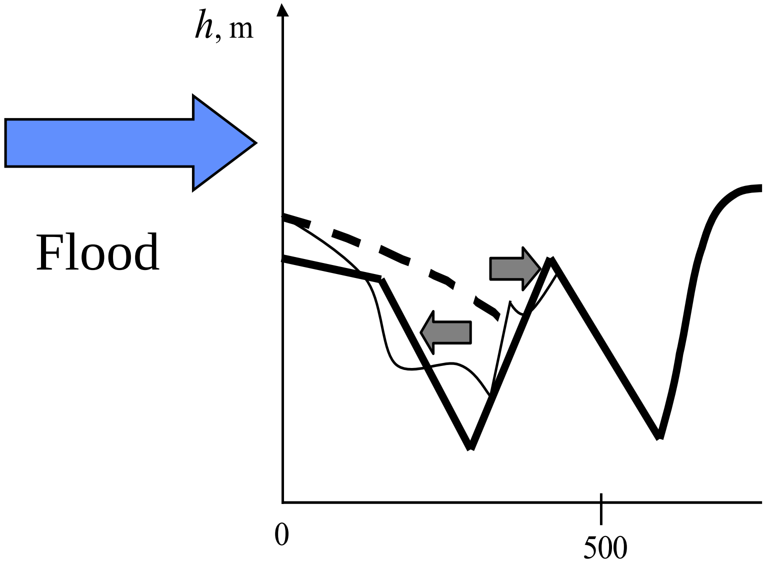
where the canal flow and the flood flow depth h_p and h_S are respectively represented by

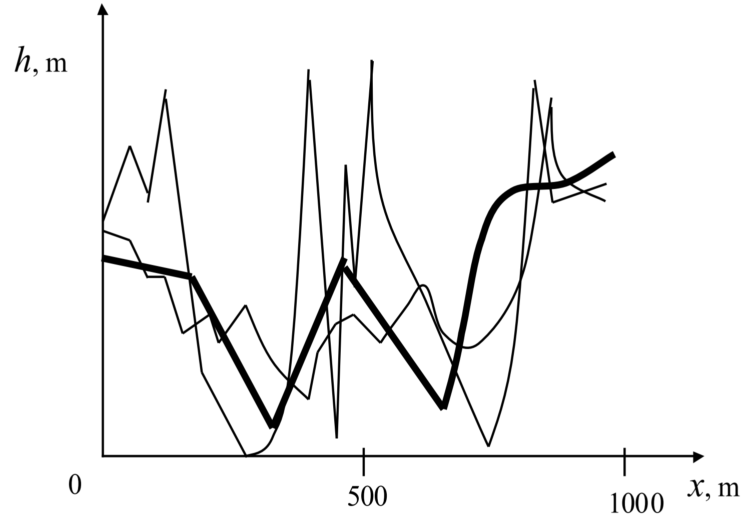
$$h_p(x,y,t) = \begin{cases} Z - z_p, & \text{if } \eta \geq Z \\ \eta - z_p, & \text{if } z_p \leq \eta \leq Z, \\ 0, & \text{otherwise.} \end{cases}$$

$$h_S(x,y,t) = \begin{cases} \eta - Z, & \text{if } \eta \geq Z, \\ 0, & \text{otherwise.} \end{cases}$$



Development of instability for a conventional numerical scheme





$$\mu \frac{\partial \eta}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial \eta}{\partial x}$$

Singularity when *h=0*

Even small violations of the positivity of the water depth induce unwanted oscillations and instability. Further analysis of non-linear degenerate parabolic systems shows that under certain conditions the equations generate breaking of waves and the shock waves.

Development of the Numerical Scheme

1) Explicit scheme ?

2) Implicit scheme ?

3) Cut the negative depth ?

4) "Wet/dry cells" ?

5) Moving boundary ?

6) Smoothing ?

Non negative numerical algorithm.1D illustration

The key idea is based on a first-order approximation of the “hyperbolic” (transport) terms by directed differences combined with the consistent right-hand (left-hand) approximations for the diffusion terms

$$\mu \frac{\partial \eta}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial \eta}{\partial x} + T$$

<u>consistent</u> right/left-hand approximations

$$\eta = h + z$$

“Diffusion” term “Hyperbolic” term

$$\mu \frac{\partial h}{\partial t} = \frac{\partial}{\partial x} D \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} D \frac{\partial z}{\partial x} + T \equiv \frac{\partial}{\partial x} D \frac{\partial h}{\partial x} + \frac{\partial}{\partial x} h d z_x + Rh$$

$$d = D/h,\ R = T/h,\ z_x = \partial z/\partial x$$

directed differences

1D Non negative numerical algorithm

$$(w_h)_k^{n+1}(\tilde{h}_k^{n+1} - h_k)/\Delta\tau = \{\tilde{d}_{k+\alpha}^{n}[\tilde{h}_{k+\alpha}^{n}(\tilde{h}_{k+1}^{n+1} - \tilde{h}_k^{n+1}) + \tilde{h}_{k+\alpha,l}^{n+1}(z_{k+1} - z_k)] - \tilde{d}_{k-1+\beta}^{n}[\tilde{h}_{k-1+\beta}^{n}(\tilde{h}_k^{n+1} - \tilde{h}_{k-1}^{n+1}) + \tilde{h}_{k-1+\beta}^{n+1}(z_k - z_{k-1})]\}/\Delta x^2$$

$$\alpha = \begin{cases} 0, & z_{k+1} - z_k < 0 \\ 1, & \text{otherwise} \end{cases}$$

$$\beta = \begin{cases} 0, & z_k - z_{k-1} < 0 \\ 1, & \text{otherwise} \end{cases}$$

2D Non negative numerical algorithm

$$(\tilde{h}_{k,l}^{n+1} - h_{k,l})/\Delta\tau = \{\tilde{d}_{k+\alpha,l}^{n}[\tilde{h}_{k+\alpha,l}^{n}(\tilde{h}_{k+1,l}^{n+1} - \tilde{h}_{k,l}^{n+1}) + \tilde{h}_{k+\alpha,l}^{n+1}(z_{k+1,l} - z_{k,l})]$$

$$-\tilde{d}_{k-1+\beta,l}^{n}[\tilde{h}_{k-1+\beta,l}^{n}(\tilde{h}_{k,l}^{n+1} - \tilde{h}_{k-1,l}^{n+1}) + \tilde{h}_{k-1+\beta,l}^{n+1}(z_{k,l} - z_{k-1,l})]\}/\Delta x^2 + \{\tilde{d}_{k,l+\gamma}^{n}[\tilde{h}_{k,l+\gamma}^{n}(\tilde{h}_{k,l+1}^{n} - \tilde{h}_{k,l}^{n+1})$$

$$+\tilde{h}_{k,l+\gamma}^{n+1-\gamma}(z_{k,l+1} - z_{k,l})] - \tilde{d}_{k,l-1+\delta}^{n}[\tilde{h}_{k,l-1+\delta}^{n}(\tilde{h}_{k,l}^{n+1} - \tilde{h}_{k,l-1}^{n}) + \tilde{h}_{k,l-1+\delta}^{n+\delta}(z_{k,l} - z_{k,l-1})]\}/\Delta y^2 + \hat{T}_{k,l},$$

where $\alpha \equiv \alpha(k,l) = \begin{cases} 0, \text{ if } & z_{k+1,l} - z_{k,l} \le 0, \\ 1, & \text{otherwise,} \end{cases}$ $\beta \equiv \beta(k,l) = \begin{cases} 0, \text{ if } & z_{k,l} - z_{k-1,l} \le 0, \\ 1, & \text{otherwise,} \end{cases}$

$$\gamma \equiv \gamma(k,l) = \begin{cases} 0, \text{ if } & z_{k,l+1} - z_{k,l} \le 0, \\ 1, & \text{otherwise,} \end{cases} \quad \beta = \beta(k,l) = \begin{cases} 0, \text{ if } & z_k - z_{k-1} \le 0, \\ 1, & \text{otherwise} \end{cases}$$

The source term is approximated by means of the following regularization

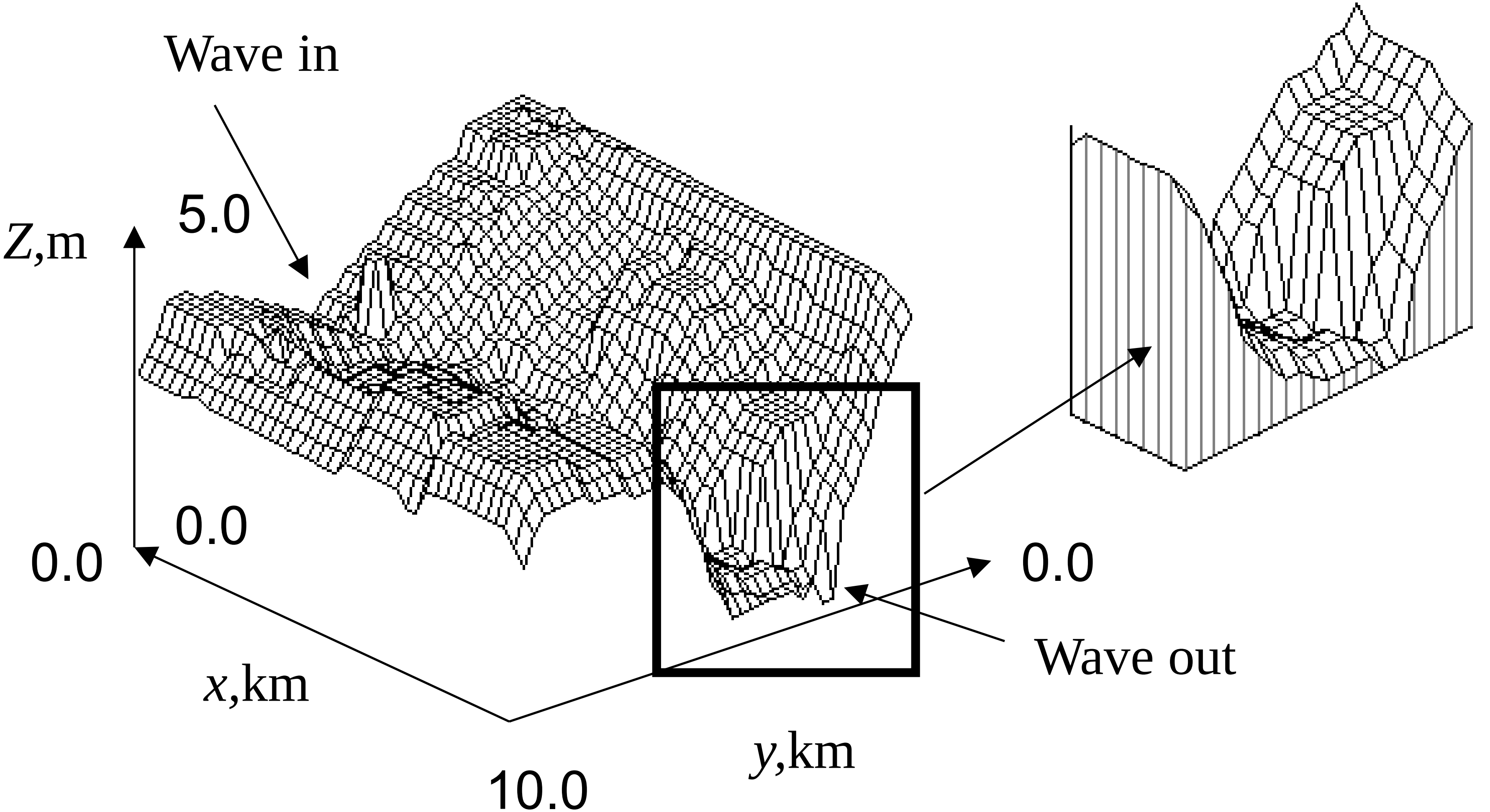
$$\hat{T}_{k,l} = \begin{cases} \tilde{T}_{k,l}^{n+1}, & \text{if } \tilde{T}_{k.l}^{n+1} > 0 \\ \tilde{T}_{k,k}^{n}\tilde{h}_{k.l}^{n+1}/\tilde{h}_{k.l}^{n}, & \text{otherwise.} \end{cases}$$

Properties of the non negative algorithm

1. The flow depth *h* is non-negative.
2. The scheme is a discrete analogy of the mass conservation law.
3. If the diffusion wave equation has a constant solution then it is the exact solution of the proposed finite difference scheme.
4. The non-negative algorithm converges.
5. Unconditional stability.

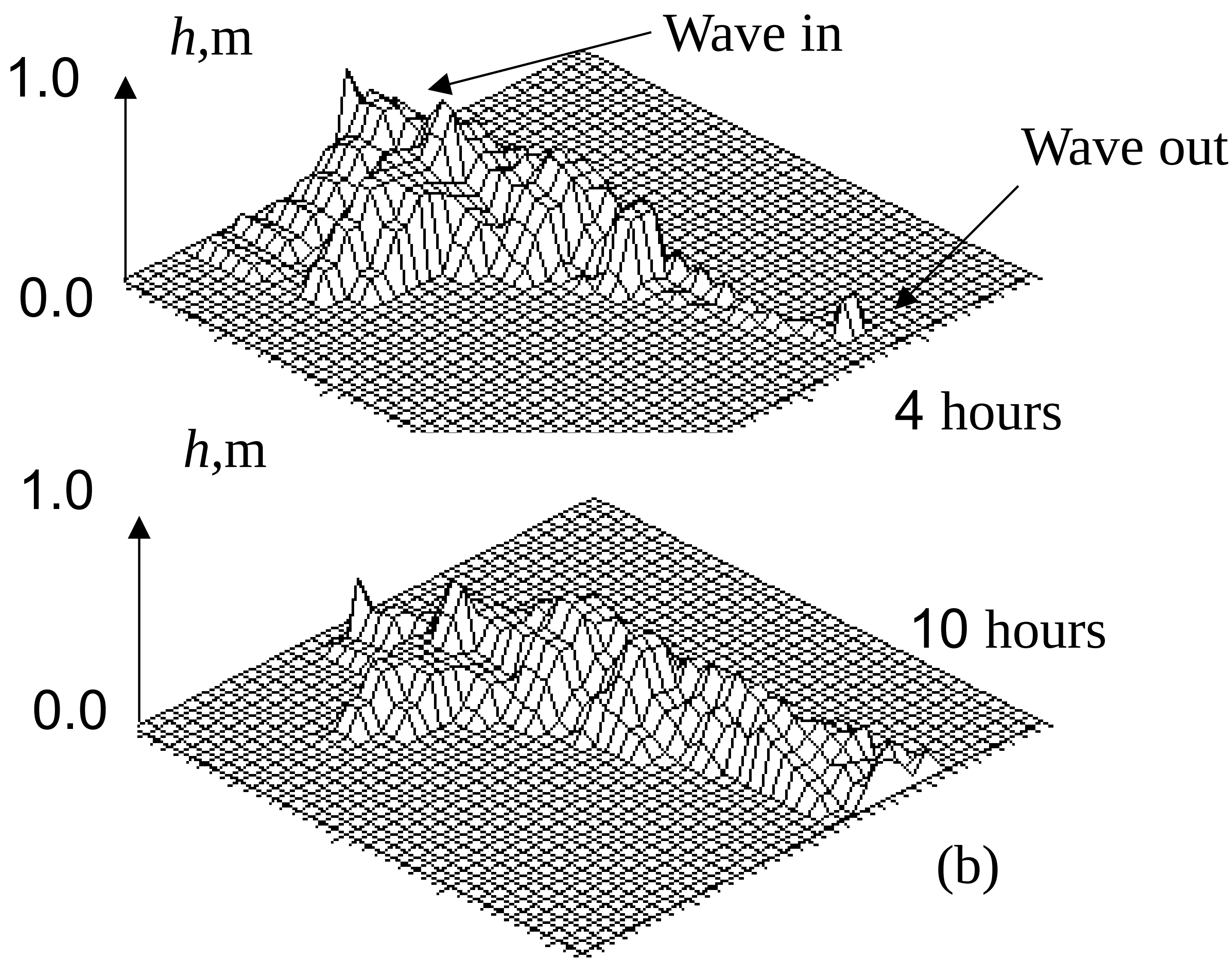
Makhanov, S.S., S. Vannakrairojn, and E.J. Vanderperre (1999). A two-dimensional numerical model of flooding in East-Bangkok, Journal of Hydraulic Engineering, Vol. 25, No. 4

Makhanov, S.S. and A.Yu. Semenov (2003). Six numerical schemes for parabolic initial boundary value problems with a priory bounded solution, Applied Numerical Mathematics, Vol. 46, pp. 331-351.



Surface wave propagation. Ground elevations.

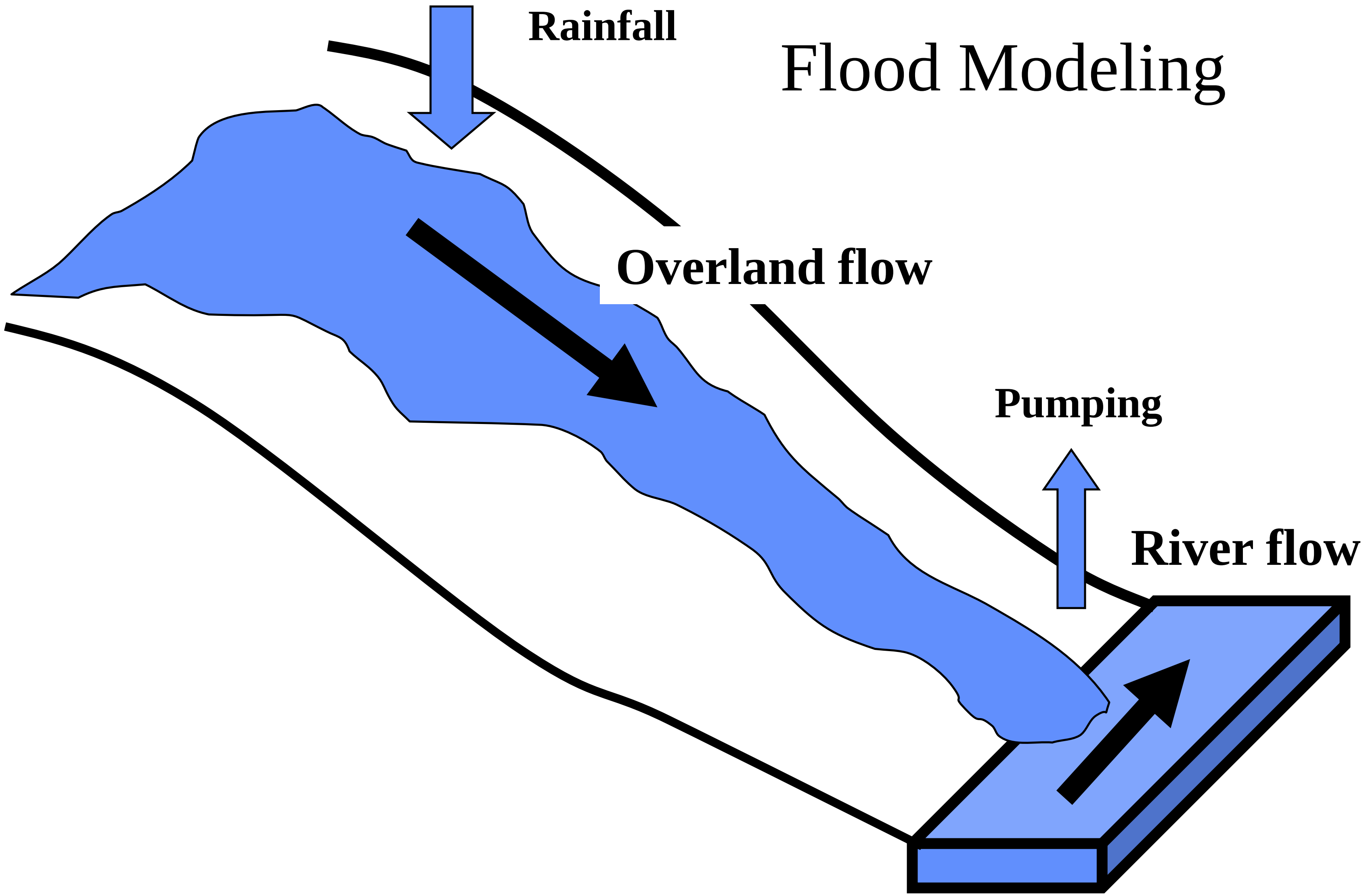
Surface wave propagation. Water Depth



Efficiency of the algorithm. Average number of iterations.

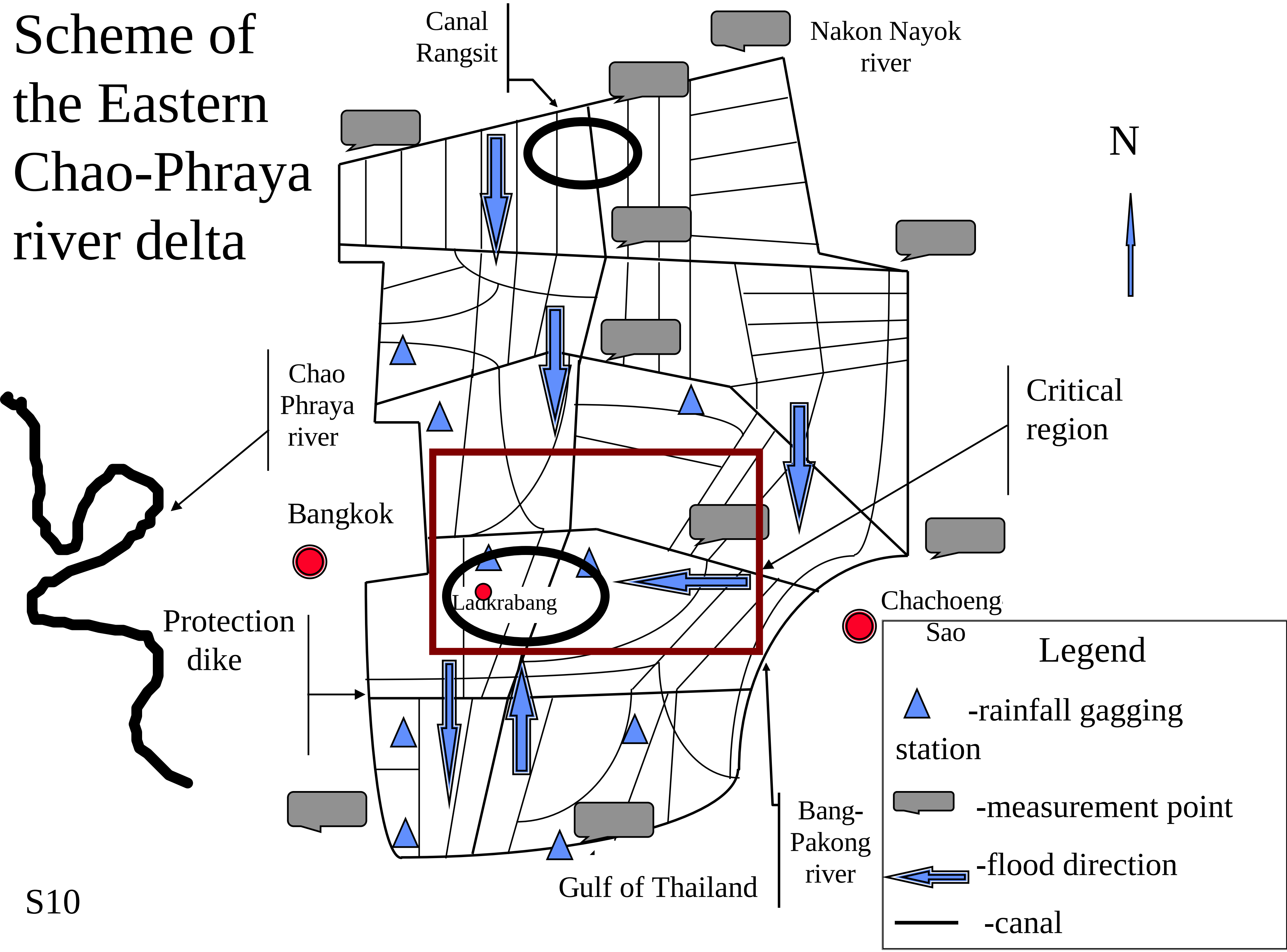
Non-negative algorithm	Implicit 2-d order Scheme	Explicit 2-d order Scheme	Time step	Spatial step
11	7/(-)	7	10000	500
21	Diverges	34	50000	500
22	Diverges	67	100000	500
23	Diverges	334	500000	500
14	12/(-)	14	10000	250
25	Diverges	67	50000	250
21	Diverges	134	100000	250
27	Diverges	667	500000	250
20	Diverges	27	10000	125
21	Diverges	134	50000	125
26	Diverges	267	100000	125
30	Diverges	1335	500000	125

Flood Modeling



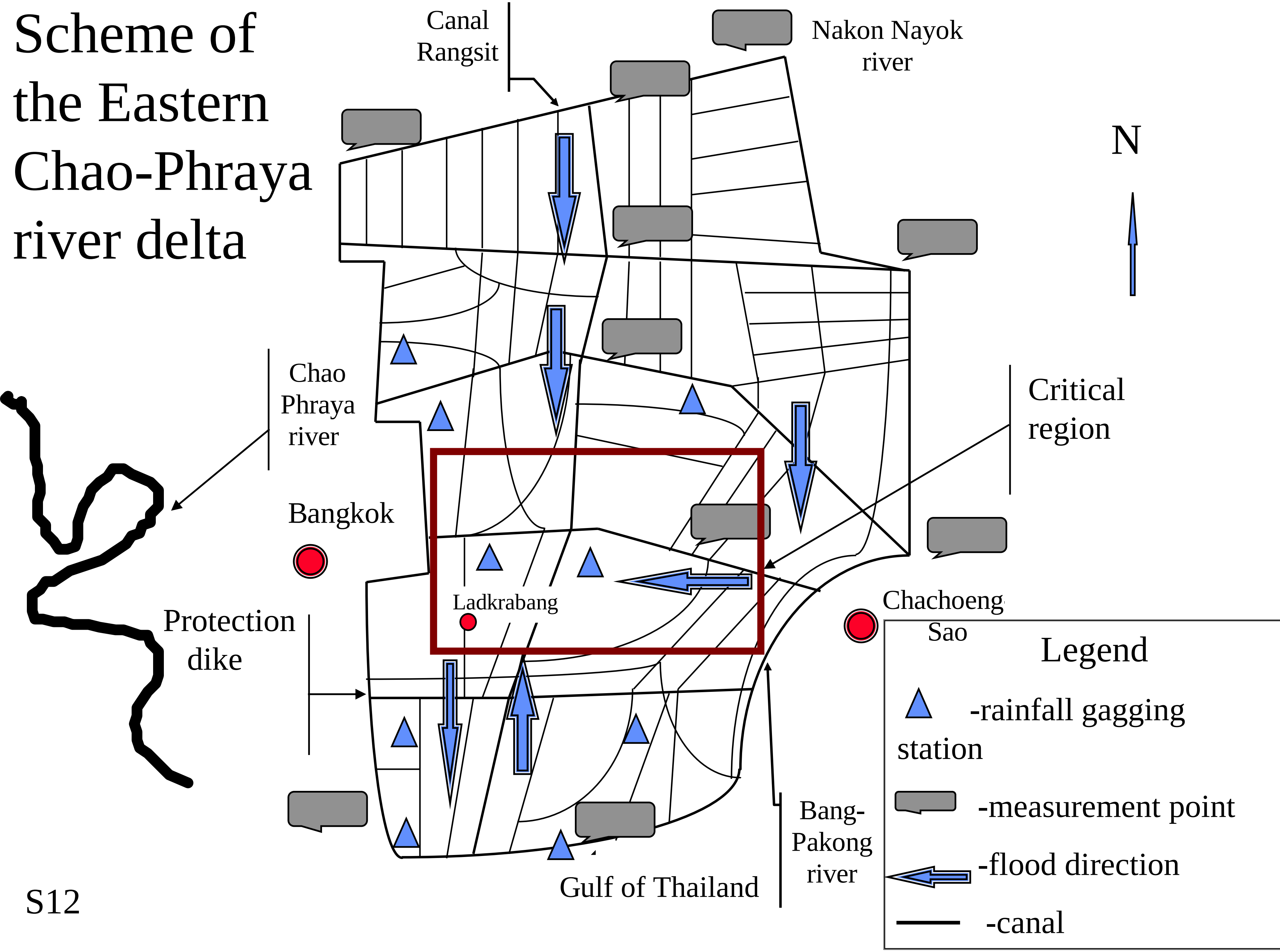
The eastern areas of Bangkok are characterized by a high density of irrigation networks located in the Chao-Phraya river delta. During the hot season (March-May), the eastern areas are preserved for water conservation. During the rainy season, the extremely low gradient of ground elevations combined with a heavy rainfall(1200-1500 mm) and a widening of the Chao-Phraya river, causes annual flooding in almost all parts of this province. The elevations of the western part are ranging from 0 to 0.5 m (with regard to the South-China sea) whereas the average level of the ground elevations is about 1.5 m. The main flood protection measures, applied to match the impact of the wet season in this area, are based on the construction of dikes, dams and pumping stations located at the boundary. The numerical modeling of the huge canal network in the delta of the Chao-Phraya river , is a formidable task. First of all, we have to perform an input by establishing a suitable link with an appropriate GIS. Secondly, the input data is incomplete and far from reliable. Moreover, the data is frequently fluctuating due to random activities of small operating farms. Finally, the water system is characterized by: 1) Complex geometries of the main and the magistrate canals. 2) High density of the irrigation network. 3) Incomplete data. 4) Wet and dry zones with irregular, time dependent boundaries. Therefore, an implementation of the conventional models, especially for a scheme comprising small canals, tends towards extremely time consuming and costly computations

Scheme of the Eastern Chao-Phraya river delta



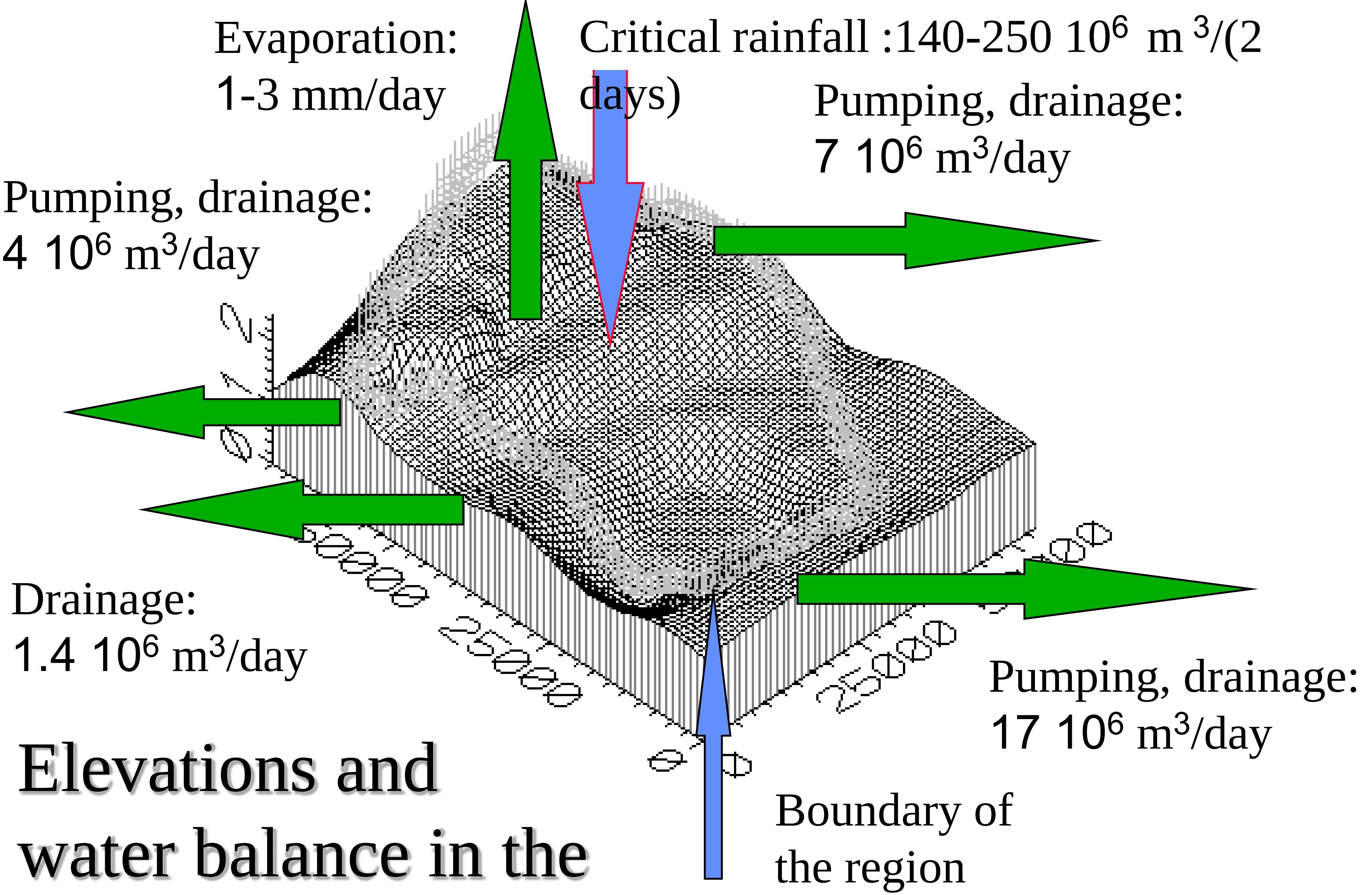
S10

Scheme of the Eastern Chao-Phraya river delta

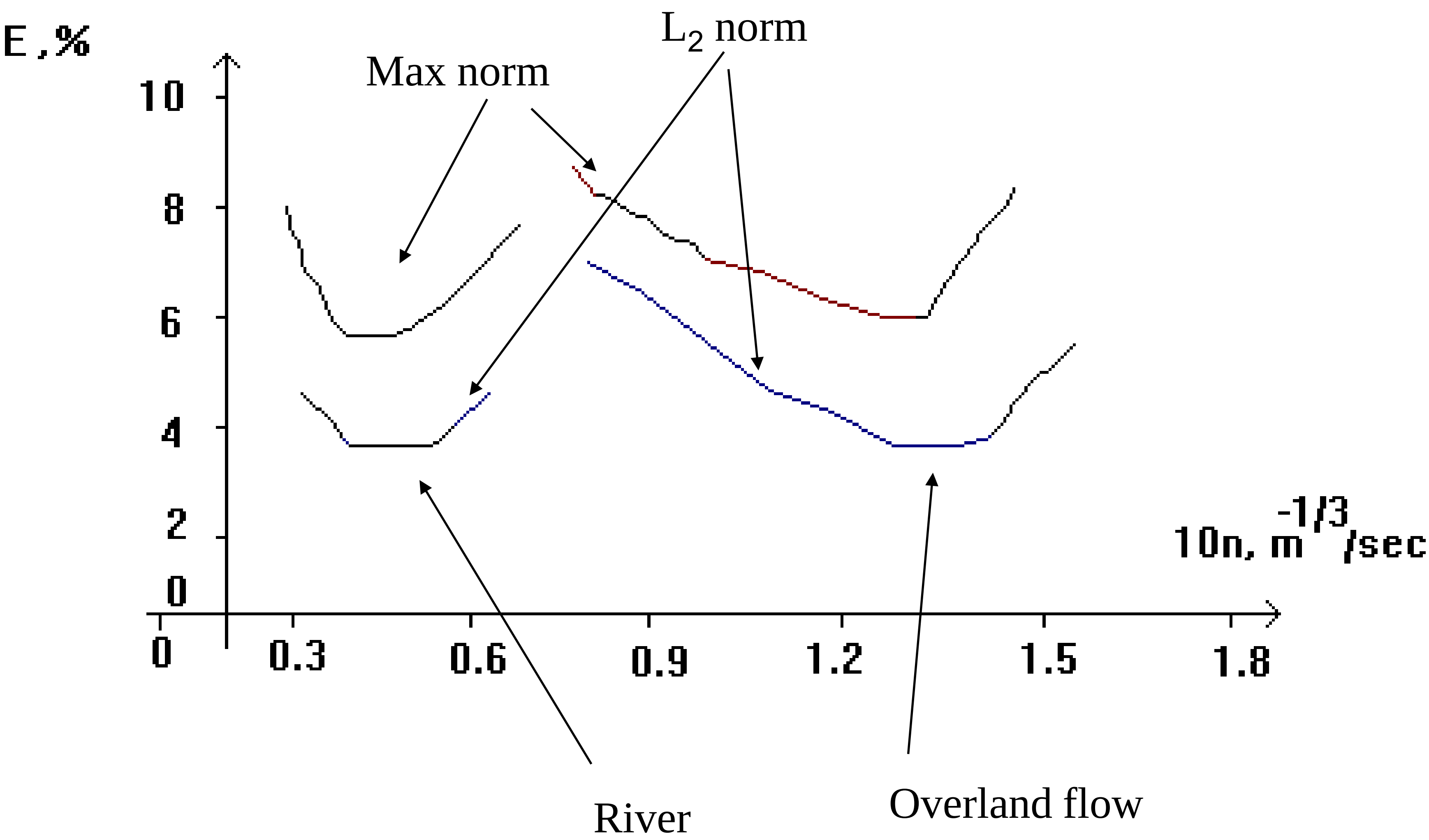


S12

Elevations and water balance in the East of Bangkok

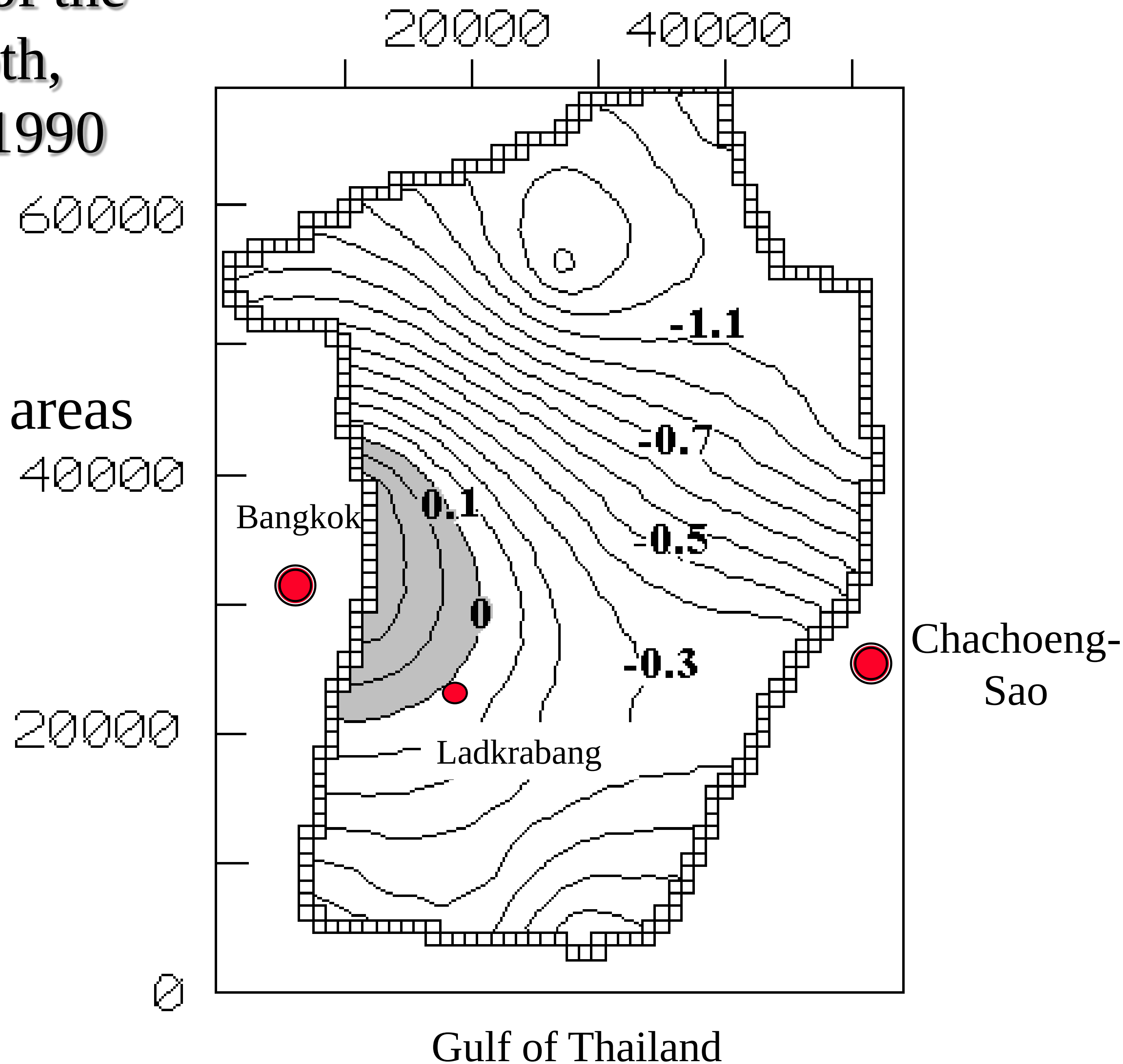


Calibration by the Manning's coefficients

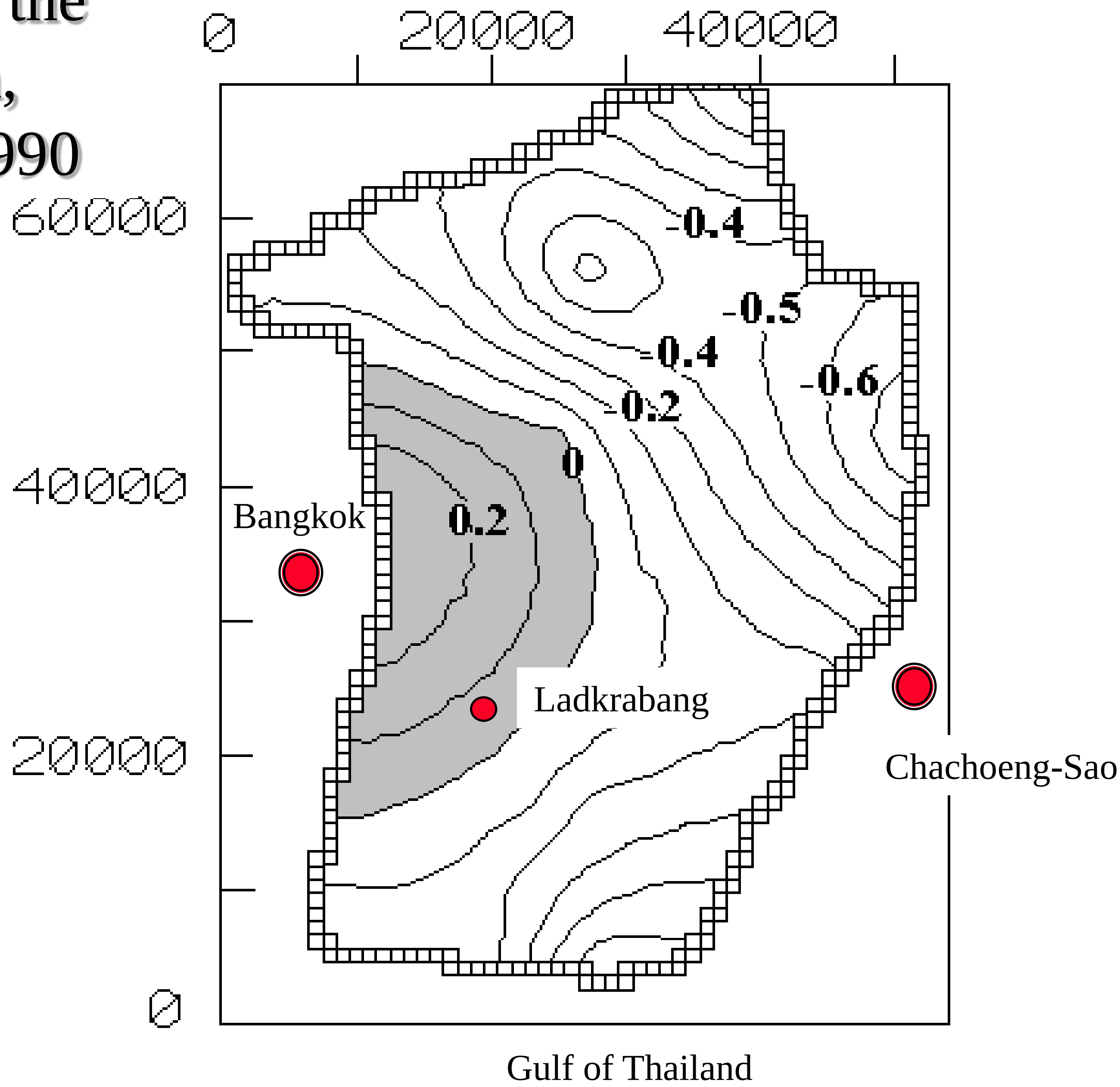


Contour lines of the flood depth, September 5,1990

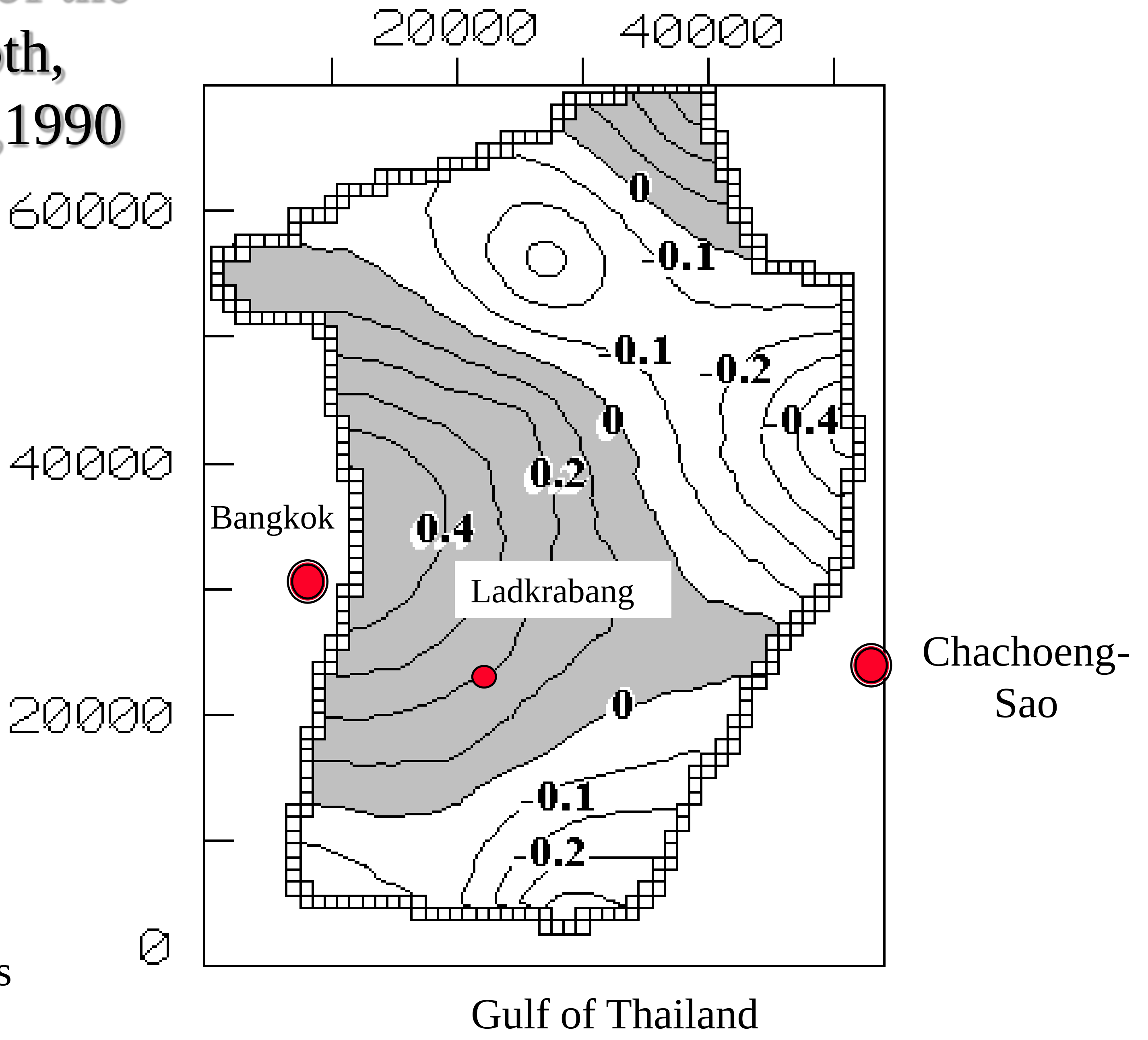
-flooded areas



Contour lines of the flood depth, September 10,1990



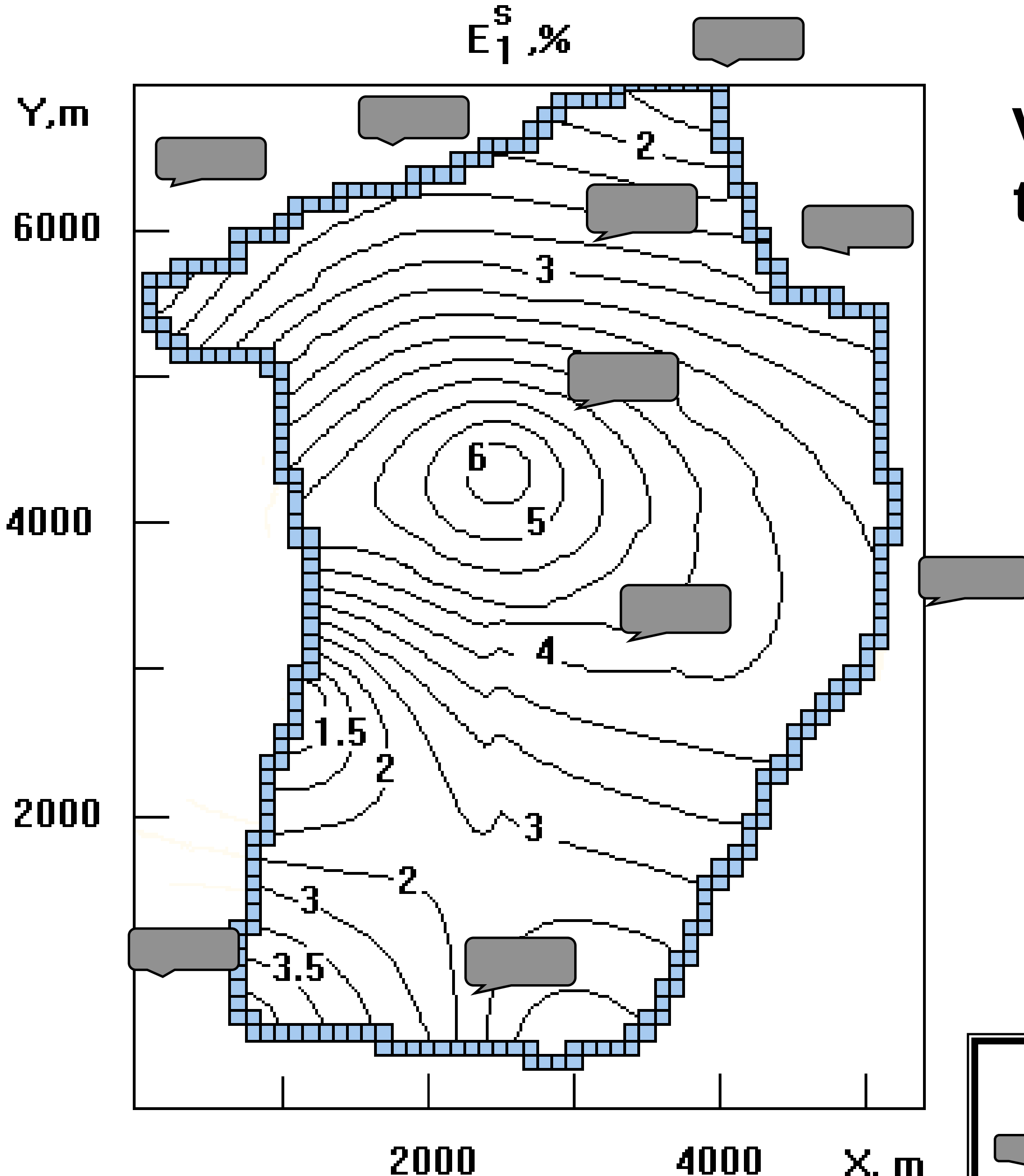
Contour lines of the flood depth, September 25,1990



- flooded areas

S16

Verification by the Mean Error



Legend

 --measurement point

S17

Comparison with other schemes

Maximum/average number of iterations

$\boldsymbol{S_1}$	S_2	$\boldsymbol{S_{12}}$	S_3	S_4	S_τ	τ,day
11/1.2	20/7(-)	**10/1.1**	27/8(-)	26/10(-)	5	0.01
12/1.3	Diverges	**15/1.2**	Diverges	Diverges	48	0.1
18/2.5	Diverges	**16/2.2**	Diverges	Diverges	471	1
21/4.1	Diverges	**19/3.2**	Diverges	Diverges	1440	2

$\boldsymbol{S_1}$-the non-negative techniques

S_2 -symmetric

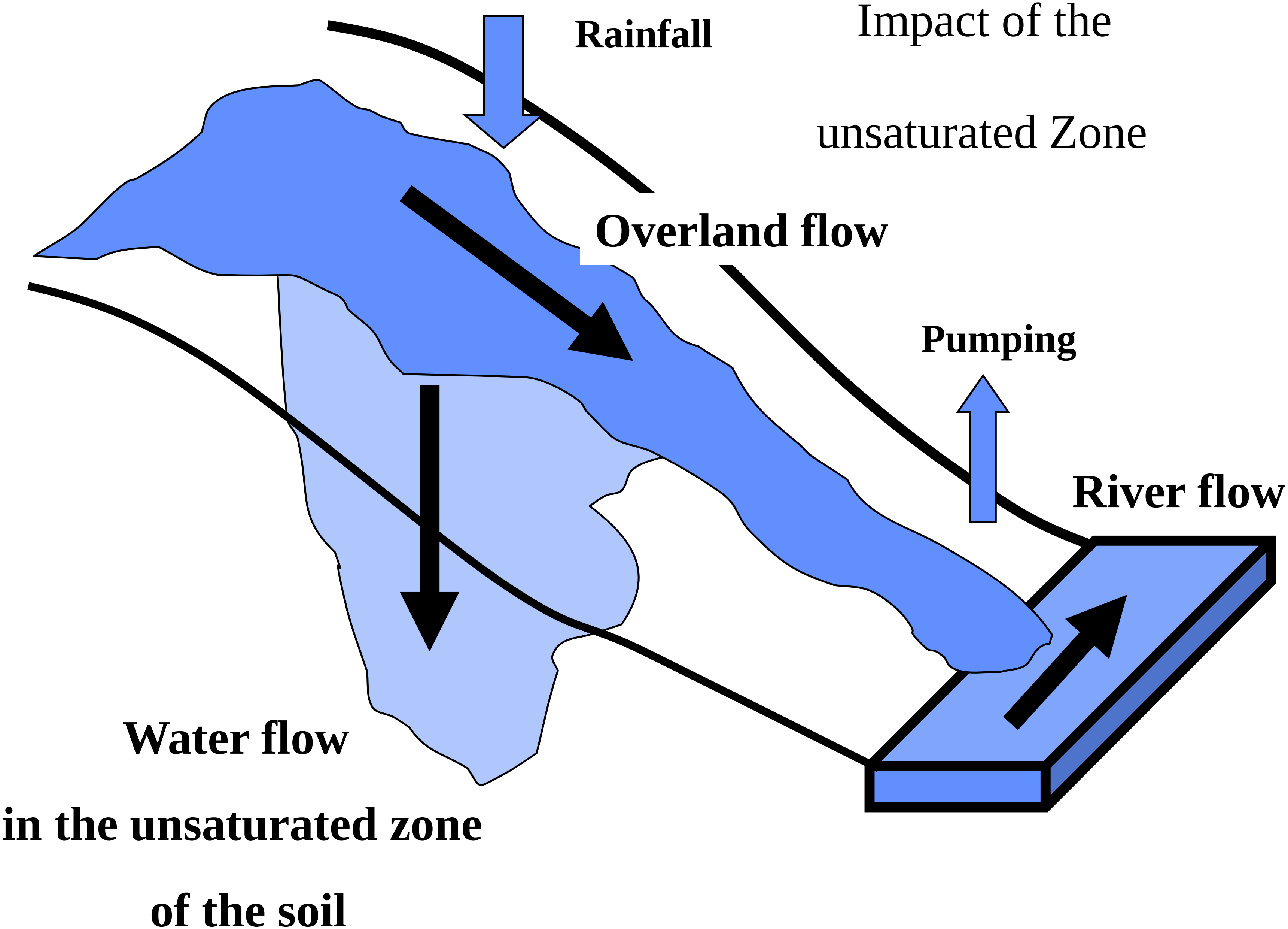
S_{12} - the hybrid: S_1/S_2

S_3-standard implicit

S_4 -"cut a negative depth-scheme"

S_τ -the standard explicit

Impact of the unsaturated Zone



S19

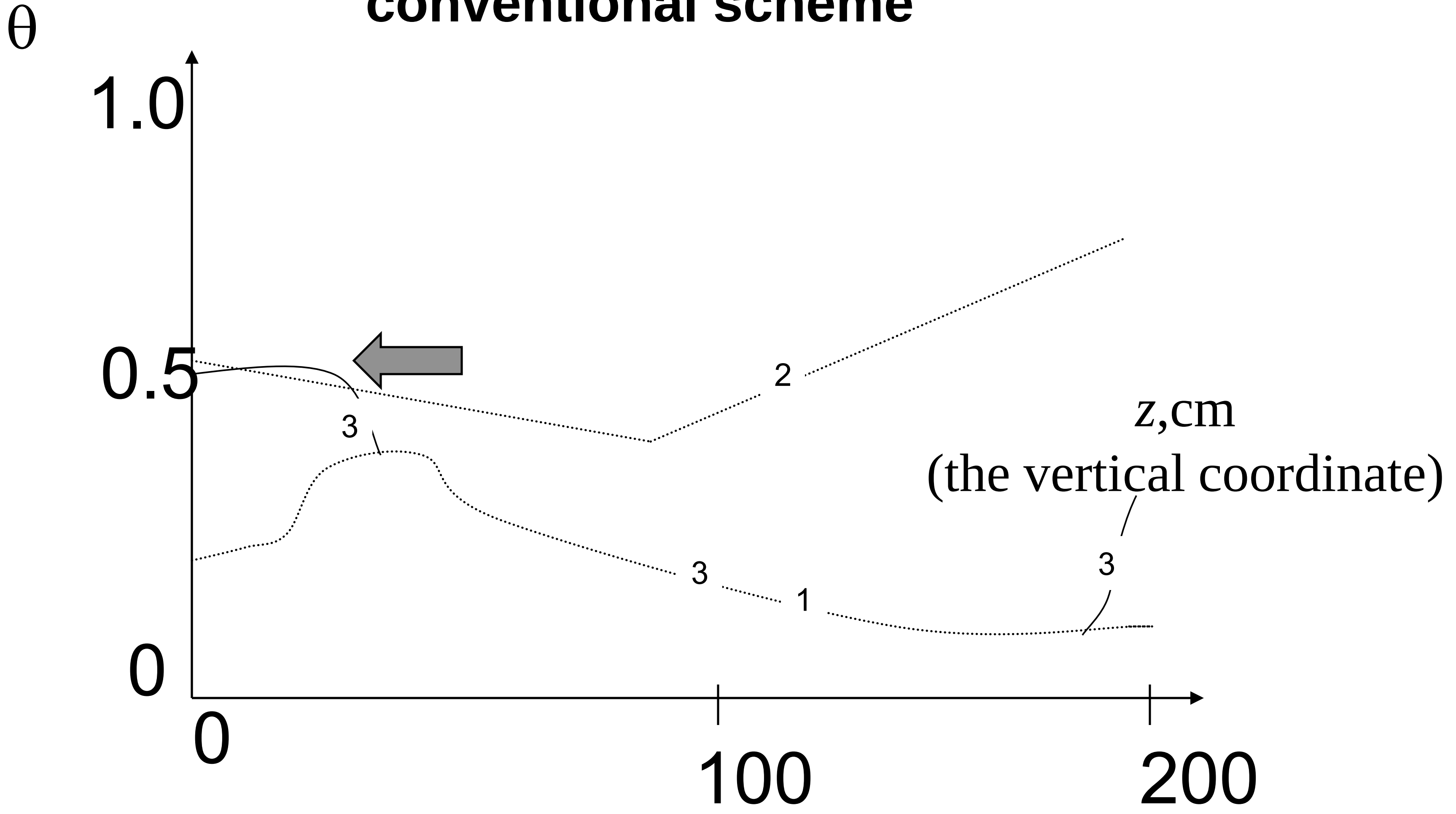
Richard’s equation

$$\frac{\partial \theta}{\partial t} = \mathrm{div}\, D \mathrm{grad}\, \theta - \frac{\partial K}{\partial \mathrm{z}} + R\,, \quad D = K \frac{\partial \psi}{\partial \theta}$$

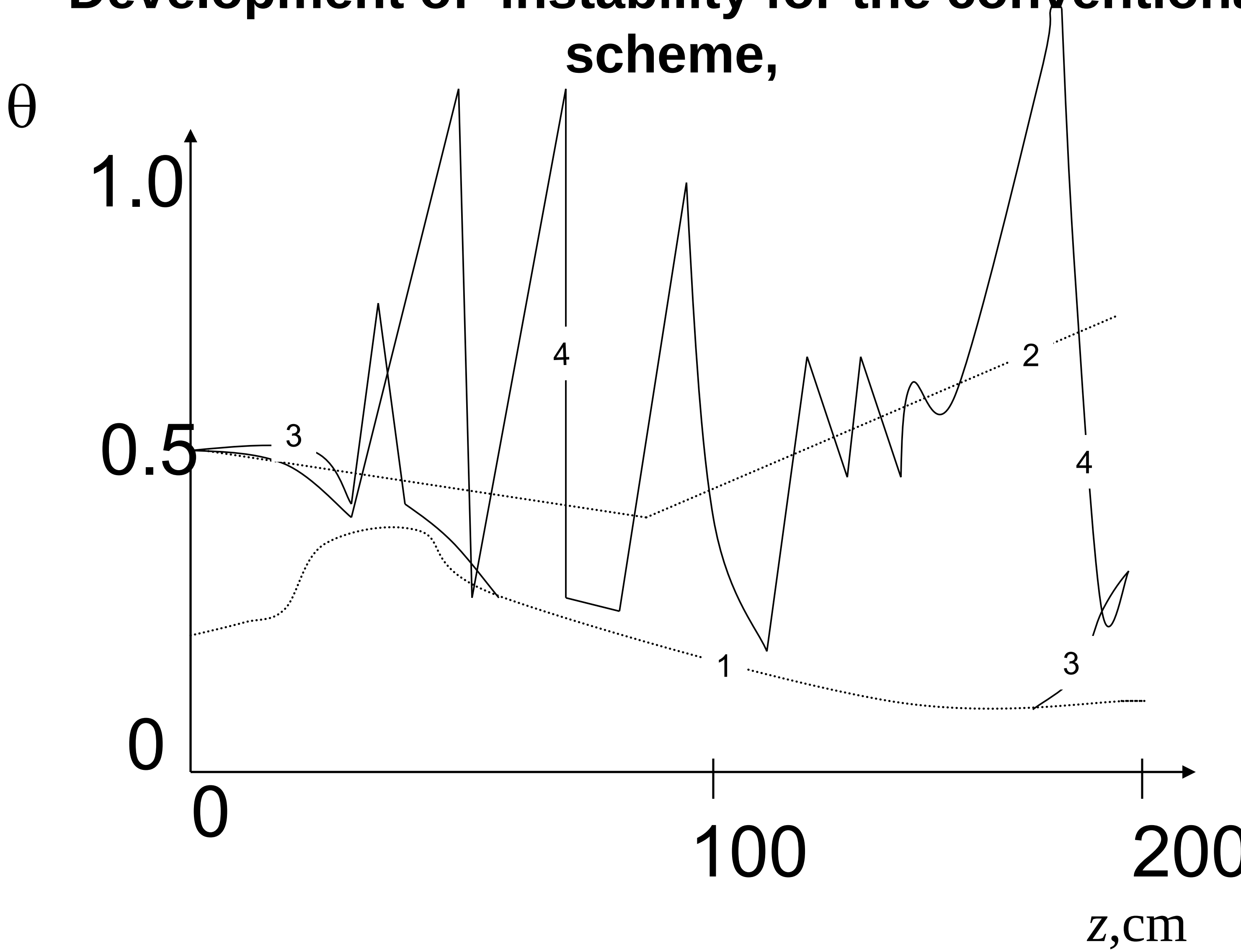
$$\theta_{\min} \le \theta \le \theta_{\max} \le 1$$

$\theta = \theta(x,z,t)$ is the soil moisture concentration,
$K=K(x,z,\theta)$ the hydraulic conductivity ,
$D=D(x,z,\theta)$ the soil water diffusivity,
$\psi=\psi(x,z,t,\theta)$ the capillary potential,
$R=R(x,z,t, \theta)$ the source (sink).
$\theta_{\min}$, $\theta_{\max}$ the residual moisture concentration and the porosity.

Numerical solution of Richard's equation. Development of instability for the conventional scheme



Numerical solution of Richard's equation. Development of instability for the conventional scheme,



S22

Numerical Algorithm for Richard's equation, 1D illustration

Step 1. Solution with with respect to $\eta = \theta - \theta_{min}$

$$\frac{\partial}{\partial z} D \frac{\partial \theta}{\partial z} - \frac{\partial K}{\partial z} = \frac{\partial}{\partial z} D \frac{\partial \eta}{\partial z} + \frac{\partial \eta a}{\partial z},$$

Step 2. Solution with with respect to $\mu = \theta_{max} - \theta$

Step 3. Correction

$$\tilde{\theta}_m = \theta_{max} + (\theta_{max} - \theta_{min})\tilde{\mu}_m / (\tilde{\mu}_m + \tilde{\nu}_m)$$

(guarantees the identity $\nu + \mu = \theta_{max} - \theta_{min}$).

Properties of the algorithm

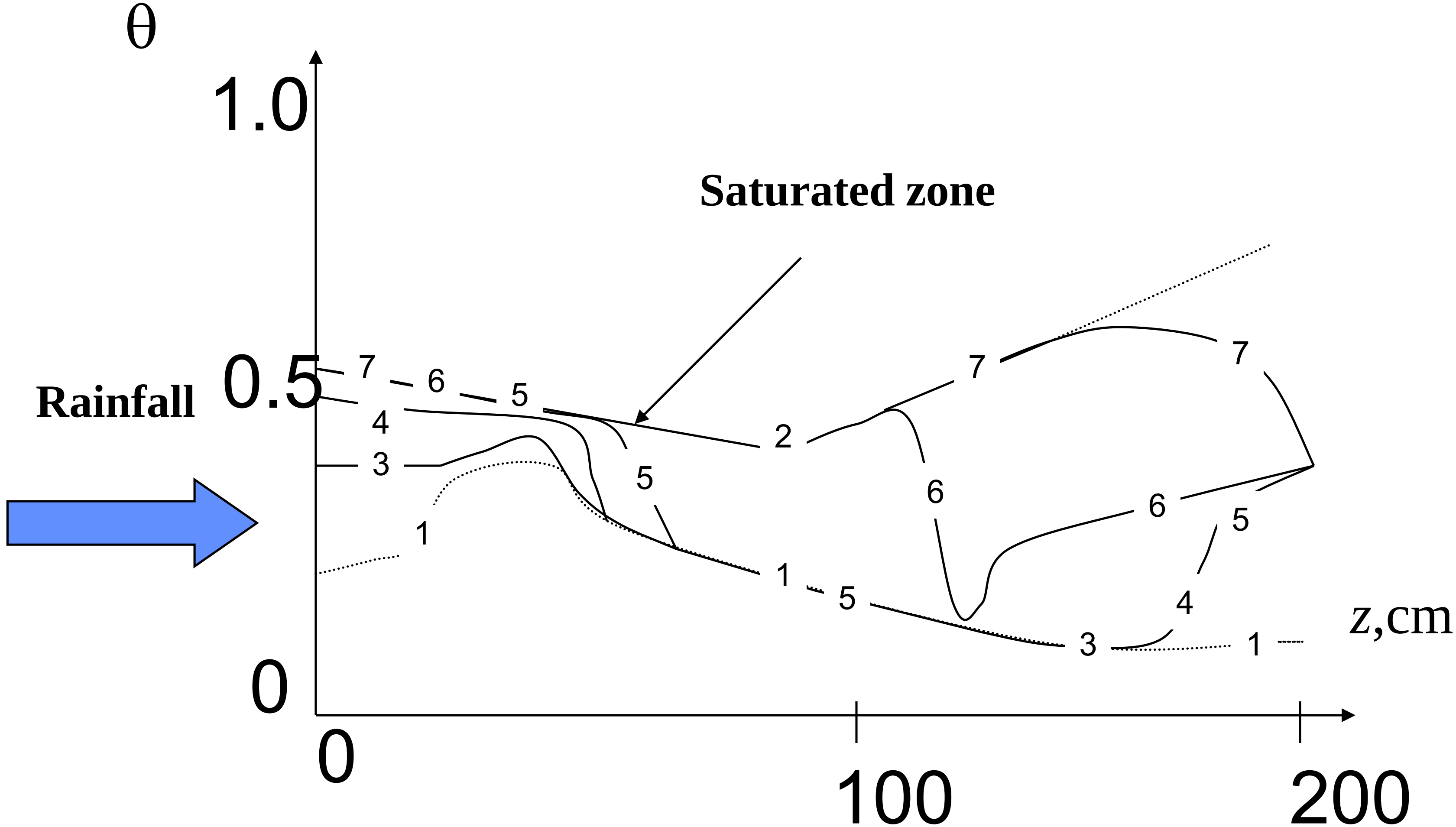
1. *The numerical solution satisfies* $\theta_{\min} \leq \tilde{\theta}_m^{n+1} \leq \theta_{\max}$

2. *The algorithm converges*

3. *The scheme is a discrete analogy of the mass conservation law*.

Makhanov, S.S. and A.Yu. Semenov (2003). Six numerical schemes for parabolic initial boundary value problems with a priory bounded solution, Applied Numerical Mathematics, Vol. 46, pp. 331-351.

Numerical solution of the infiltration problem

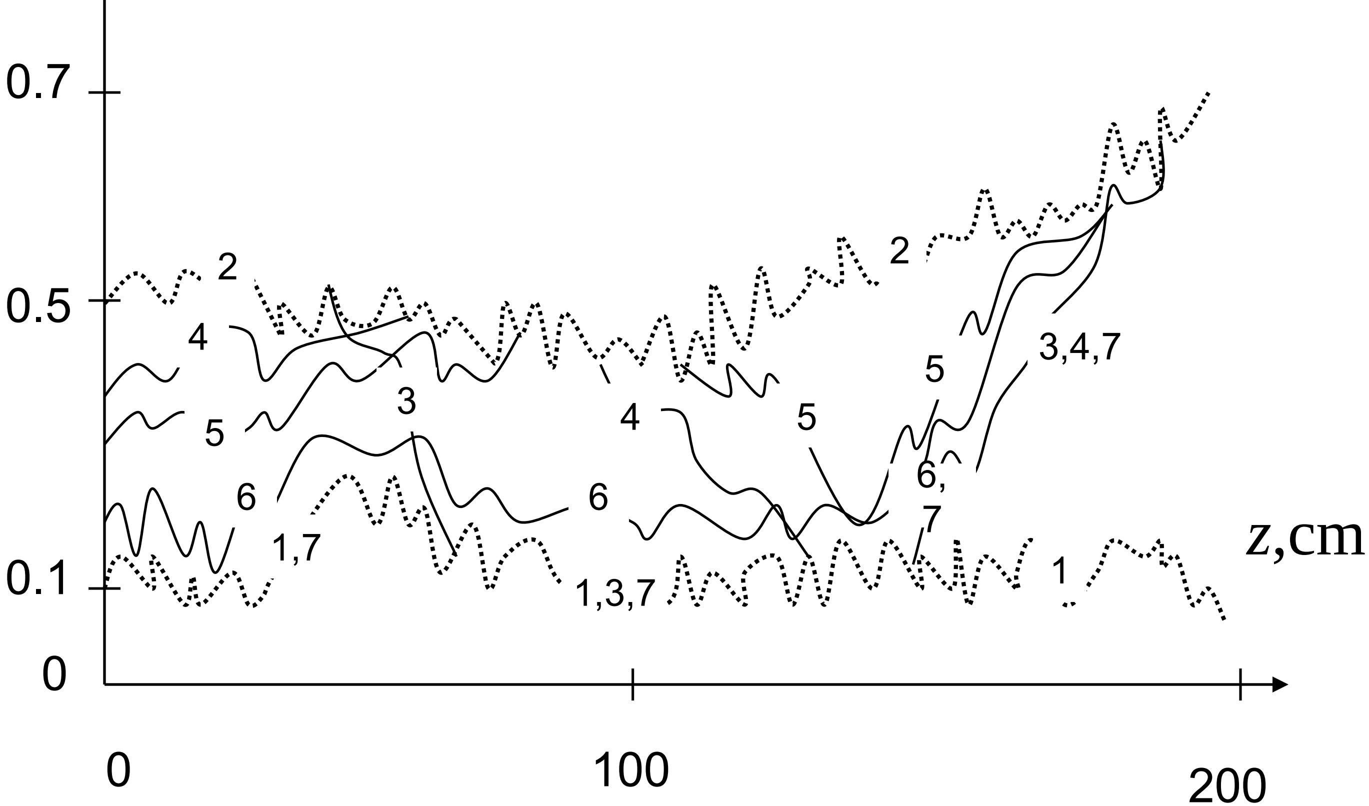


(1)-θ_{min}, (2)- θ_{max} , (3) θ (z, t), *t* =0.25h, (4) θ (z, t), *t* =0.5h, (5)- *t* =2h, (6)- *t* =4h, (7) *t* =6 h .

Efficiency of the proposed algorithm

S_1	S_2	S_τ	τ, sec	Δ, cm
8/6	4/3(-)	0.11	10	10
8/7	5/4(-)	0.22	10	5
13/9	13/11	0.43	10	2.5
20/12	20/12	0.88	10	1.25
10/9	9/7(-)	0.55	50	10
13/11	Diverges	1.1	50	5
20/13	Diverges	2.15	50	2.5
28/21	Diverges	4.4	50	1.25
22/16	Diverges	11	1000	10
30/21	Diverges	22	1000	5
46/27	Diverges	44	1000	2.5
72/30	Diverges	88	1000	1.25
46/20	Diverges	110	10000	10
65/28	Diverges	220	10000	5
100/46	Diverges	430	10000	2.5
154/65	Diverges	880	10000	1.25

Saturation wave. The numerical solution in the case of randomly perturbed bounds



$$\theta_{\max}(z) = \bar{\theta}_{\max}(z) + \Theta_{\max},\ \theta_{\min}(z) = \bar{\theta}_{\min}(z) + \Theta_{\min}$$

(1)- $\theta_{\min}$, (2)- $\theta_{\max}$,(3) $\theta(z,t)$, t=2h (3) t=3h (4) t =4h (5) t=5h (6) t=6h

Efficiency of the proposed algorithm

S_1	S_2	S_τ	τ, sec	Δ, cm
12/8	3/2(-)	0.05	10	10
18/10	5/3(-)	0.1	10	5
26/12	7/5	0.2	10	2.5
36/20	14/7	0.4	10	1.25
18/10	10/6(-)	0.25	50	10
19/11	Diverges	0.5	50	5
46/14	Diverges	1.1	50	2.5
48/22	Diverges	2.2	50	1.25
30/10	Diverges	4.7	1000	10
44/19	Diverges	8.3	1000	5
64/21	Diverges	16.3	1000	2.5
98/25	Diverges	32.6	1000	1.25
40/14	Diverges	41	10000	10
58/18	Diverges	81	10000	5
90/25	Diverges	163	10000	2.5
138/30	Diverges	326	10000	1.25

Conclusions

1. We proposed and analyzed a new family of **unconditionally stable numerical algorithms** to solve parabolic boundary value problems endowed with constraints.

2. The methods are applicable to simulate and **overland flows** and **unsaturated porous medium flows**.

3. The methods demonstrate an **overall priority with regard to conventional schems** when the solution approaches the prescribed boundaries.

4. The methods applied to **real world applications** provide uniform treatment of the "dry" and "wet" cells

5. The methods are in particular suitable to simulate the water systems comprising **large canal networks.**